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A Basic Lemma on Social Welfare Functions : Derivation of Arrow's and Sen's Theorems

Koji TAKAMIYA *

This paper provides a lemma from which Arrow's and Sen's theorems (Arrow, 1951, 1963 ; Sen, 1970), fundamental results of social choice theory, can be derived. The lemma gives a sufficient condition for a social welfare function to generate cyclic social preferences. Sen's theorem immediately follows from this lemma. And using an additional lemma (a variant of the result by Wilson (1972)) which connects the concepts of social welfare functions and simple games, I prove Arrow's theorem. The argument explicitly depends on the theory of simple games. Intuitively, the method of proofs shows that these famous results share a 'common source'.

JEL Classfication : C71, D71.

0. Preliminaries

Denote by N the *finite* set of individuals. Assume that N is the initial n segment $\{1, 2, \ldots, n\}$ of the natural numbers $\mathbf{N}$. Denote by X the set of alternatives with $|X| \geq 3$. X may be finite or infinite. Let B be the set of all binary relations on X satisfying for any $P \in B$,[1]

(Asymmetry) $\quad \forall x, y \in X,\ xPy \Rightarrow \neg yPx.$

Let A denote the subset of B satisfying for any $P \in A$,

(Acyclicity) $\forall m \in \mathbf{N}, \forall x_1, x_2, \ldots, x_m \in X,\ x_2Px_1, x_3Px_2, \ldots, x_mPx_{m-1} \Rightarrow \neg x_1Px_m.$

Then A is the set of *acyclic orders*. Denote by Q be the subset of B satisfying for any $P \in Q$,

(Transitivity) $\quad \forall x, y, z \in X,\ (xPy \text{ and } yPz) \Rightarrow xPz.$

Then Q is the set of *quasi orders*. Let W be the subset of B satisfying for any

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1 Throughout the paper, '¬' means negation.

$P \in W$,

(Negative Transitivity) $\forall x, y, z \in X$, $(\neg xPy$ and $\neg yPz) \Rightarrow \neg xPz$.

Then W is the set of *weak orders*. Note that $W \subset Q \subset A \subset B$. For any subset B' of B, B'^N denotes the $|N|$-fold Cartesian product of B'.

Let D be a nonempty subset of W^N. A *social welfare function* (SWF) on D is a single-valued function $F: D \rightarrow B$. On the domain D of a SWF, throughout the paper, I assume the following:

(Unrestricted Domain) $D = W^N$.

Call a SWF *acyclic* if $F(D) \subset A$. F is called *cyclic* if not acyclic.

I introduce *simple games* (Shapley, 1962 ; Nakamura 1975). A simple game is a game-theoretic concept which is closely related to the concept of SWF (see e. g. Wilson, 1972 ; Peleg, 1978 ; Moulin, 1988). A simple game is a list $(N, \mathcal{W})$. Here N is the finite set of players. And $\mathcal{W}$ is the class of *winning coalitions*. $\mathcal{W}$ is a subset of $2^N \setminus \{\phi\}$ which is monotonic, i. e., satisfies

(Monotonicity) $(S \in \mathcal{W}$ and $S \subset T) \Rightarrow T \in \mathcal{W}$.

Call $\mathcal{W}$ *proper* if it satisfies

(Properness) $S \in \mathcal{W} \Rightarrow N \setminus S \notin \mathcal{W}$.

Call $\mathcal{W}$ *strong* if it satisfies

(Strongness) $(S \notin \mathcal{W}$ and $S \subset N) \Rightarrow N \setminus S \in \mathcal{W}$.

Call $\mathcal{W}$ *weak* if it satisfies

(Weakness) $\cap \mathcal{W} \neq \phi$.

Let $(N, \mathcal{W})$ be a simple game which is not weak. The *Nakamura number* (Nakamura, 1979) $\mathcal{V}(N, \mathcal{W})$ of $(N, \mathcal{W})$ is defined by

$$\mathcal{V}(N, \mathcal{W}) = \min \{|\sigma| \mid \sigma \subset \mathcal{W},\ \sigma \neq \phi,\ \cap \sigma = \phi\}.$$ [2]

2 Nakamura (1979) proved important theorems on the relation between the existence of core of simple games and the Nakamura number. But I do not introduce those results here.

Let a SWF F be given. Then I define a set-valued function $\mathscr{D}$ associated with F as follows :

$$\mathscr{D} : (X\times X)\setminus\{(x,x)\mid x\in X\}\to\to 2^N\setminus\{\phi\} \text{ such that}$$
$$\forall x,y\in X, x\neq y, [S\in\mathscr{D}(x,y)\Leftrightarrow [\forall P\in D, (\forall i\in S, xP^iy)\Rightarrow xF(P)y]]$$

As usual, if $S\in\mathscr{D}(x,y)$, I call the coalition S *decisive* over (x,y).
Denote by $\mathscr{W}_F$ the set

$$\cap\{\mathscr{D}(x,y)\mid x,y\in X, x\neq y\}.$$

I call an element of $\mathscr{W}_F$ a *winning coalition*. Since $\mathscr{W}_F$ is monotonic, $(N,\mathscr{W}_F)$ forms a simple game. By definition (since any value of F is asymmetric), $\mathscr{W}_F$ is proper.
Finally, I introduce some properties of SWFs F.

Unanimity (UN) : $\forall x,y\in X, (\forall i\in N, xP^iy)\Rightarrow xF(P)y.$

Since $\mathscr{W}_F$ is monotonic, F satisfies (UN) if and only if $\mathscr{W}_F\neq\phi$

Independence of *Irrelevant Alternatives* (IIA)[3] :
$\forall x,y\in X, x\neq y, \forall P,Q\in D, P\mid_{\{x,y\}}=Q\mid_{\{x,y\}}\Rightarrow F(P)\mid_{\{x,y\}}=F(Q)\mid_{\{x,y\}}.$

Minimal Liberalism (ML) :
$\exists S\subset N: \mid S\mid=2\ \&\ [\forall i\in S : \exists x,y\in X, x\neq y : \{i\}\in\mathscr{D}(x,y)\cap\mathscr{D}(y,x)].$

1. Results

The following lemma is the main result of the paper. The lemma gives a sufficient condition for a SWF to be cyclic (thus a necessary condition for F to be acyclic).

Lemma 1 - 1 : Let F be a SWF. Let $x_1,\ldots,x_p$ be p distinct elements of X. Then if there exist p (not necessarily distinct) nonempty subsets $S_1,\ldots,S_p$ of N satisfying

$$\forall k\in\{1,\ldots,p\}, S_k\in\mathscr{D}(x_k,x_{k+1}),$$

where $x_{p+1}=x_1$, and

$$\cap\{S_1,\ldots,S_p\}=\phi,$$

then F is cyclic.

3 " | " denotes restriction.

I prove this lemma along the same line as the proof of Theorem 10. 17 in Okada (1996, p. 328), which is a simplified version of Theorem 2.3 in Nakamura (1979).

Proof : Choose $x_1, \ldots, x_p \in X$ and $S_1, \ldots, S_p \subset N$ satisfying the above. Denote by $\mathscr{S}$ the class $\{S_1, \ldots, S_p\}$. Since $\cap \mathscr{S} = \phi$, there exists a function $\tau : N \to \{1, \ldots, p\}$ such that $\forall i \in N, i \notin S_{\tau(i)} \in \mathscr{S}$. Then choose $P \in D$ satisfying

$$\forall i \in N, \tau(i) = k \Rightarrow [x_{k+1} P^i x_{k+2}, \ldots, x_{p-1} P^i x_p, x_1 P^i x_2, \ldots, x_{k-1} P^i x_k].$$

Note that by the above definition, if $\tau(i) \neq k \in \{1, \ldots, p\}$, then $x_k P^i x_{k+1}$. Thus

$$\forall k \in \{1, \ldots, p\}, \forall i \in S_k, x_k P^i x_{k+1}.$$

Since each S_k is decisive over (x_k, x_{k+1}), I have

$$x_1 F(P) x_2, x_2 F(P) x_3, \ldots, x_{p-1} F(P) x_p, x_p F(P) x_1.$$

That is, $F(P)$ is cyclic. ☺

Next, I present a lemma which gives a sufficient condition for a coalition to be winning. This is a variant of the result obtained by Wilson (1972, Theorem 2).

Lemma 1 - 2 : Let F satisfy (IIA), (UN) and $F(D) \subset W$. Then $[\exists S \subset N, S \neq \phi : \exists P \in D : (\forall i \in S, x P^i y) \& (\forall j \in N \backslash S, y P^j x) \& \neg y F(P) x]$ $\Rightarrow S \in \mathscr{W}_F$.

Proof : The following proof depends on a standard argument, thus it suffices to sketch it. Let F be a SWF satisfying (IIA), (UN) and $F(D) \subset W$. Suppose that some $S \subset N, S \neq \phi$ and $P^* \in D$ satisfies

$$(\forall i \in S, x P^{*i} y) \& (\forall j \in N \backslash S, y P^{*j} x) \& \neg y F(P^*) x.$$

Let $z \in X \backslash \{x, y\}$. (Recall that $|X| \geq 3$.) Choose $P \in D$ which satisfies

$$P|_{\{x,y\}} = P^*|_{\{x,y\}}, (\forall i \in N, y P^i z), \text{and } (\forall i \in S, x P^i z).$$

No restriction is put on the preferences of $i \in N \backslash S$ between x and z. Then by (IIA), $\neg y F(P) x$. And by (UN), $y F(P) z$. Since $F(\bullet)$ is negatively transitive, $x F(P) z$. Thus (IIA) implies $S \in \mathscr{D}(x, z)$. By repeating similar arguments, I have $S \in \mathscr{D}(z, y)$, $S \in \mathscr{D}(x, y)$, etc. For any $v, w \in X \backslash \{x, y\}$ with $v \neq z$, essentially the same construction applies to obtain $S \in \mathscr{D}(v, w)$. Eventually, I have $S \in \mathscr{W}_F$.

2. Arrow's and Sen's Theorems

In this section, I derive two fundamental results, namely Arrow's (Arrow, 1951, 1963) and Sen's (Sen, 1970) theorems, from the lemmas proved above. Firstly, Sen's theorem follows almost immediately from Lemma 1 - 1.

Theorem 2 - 3 (Sen, 1970) : Let F be a SWF. Then if F satisfies (ML) and

(UN), then F is cyclic.

Proof : Let a SWF F satisfy (ML) and (UN). Let $k, l \in N$ with $k \neq l$. Assume that $k \in N$ is decisive over (x,y) and (y,x) and that $l \in N$ is decisive over (v,w) and (w,v). Suppose that $\{x,y\} \cap \{v,w\} = \phi$. Then since (UN) is also satisfied, $\{k\} \in \mathscr{D}(x,y), N \in \mathscr{D}(y,v), \{l\} \in \mathscr{D}(v,w)$ and $N \in \mathscr{D}(w,x)$. Clearly $\cap \{\{k\}, \{l\}, N\} = \phi$. Thus by Lemma 1 - 1, F is cyclic. For the case that $\{x,y\} \cap \{v,w\} \neq \phi$, a similar argument applies. ☺

One may regard Lemma 1 - 1 as a generalization of Sen's theorem.
Next, I derive Arrow's theorem.

Theorem 2 - 4 (Arrow, 1963) : Let F be a SWF. If F satisfies (IIA), (UN) and $F(D) \subset W$, then there exists $d \in N$ such that $\{d\} \in \mathscr{W}_F$.

I start with the following corollary of Lemma 1 - 2.

Corollary 2 - 5 : Let F a SWF. If F satisfies (IIA), (UN) and $F(D) \subset W$, then $\mathscr{W}_F$ is strong.

Proof : Let F be a SWF satisfying (IIA), (UN) and $F(D) \subset W$. Let $S \subset N$. If $S = \phi$, then by assumption $S \notin \mathscr{W}_F$, and by (UN) $N \backslash S \in \mathscr{W}_F$. Contrarily, if $S = N$, then $S \in \mathscr{W}_F$ and $N \backslash S \notin \mathscr{W}_F$. Suppose $S \neq \phi, N$. Let $x, y \in X$ with $x \neq y$. Choose $P \in D$ such that

$$(\forall i \in S, xP^i y) \text{ and } (\forall i \in N \backslash S, yP^i x).$$

Since F (●) is asymmetric, I have $\neg yF(P)x$ or $\neg xF(P)y$. If the former holds, then Lemma 1 - 2 implies $S \in \mathscr{W}_F$. Otherwise, I have $N \backslash S \in \mathscr{W}_F$. After all, for $S \subset N$, it holds $S \in \mathscr{W}_F$ or $N \backslash S \in \mathscr{W}_F$, i. e., $S \notin \mathscr{W}_F \Rightarrow N \backslash S \in \mathscr{W}_F$. ☺

To complete the derivation, I use a slight modification of Proposition 3.2 in Ishikawa and Nakamura (1979). The proof is due to Ishikawa and Nakamura, and Peleg (1978).

Lemma 2 - 6 : Let $(N, \mathscr{W})$ be a simple game. Let $\mathscr{W}$ be proper and strong. Then if there does not exist $d \in N$ such that $\{d\} \in \mathscr{W}$, then $\mathscr{V}(N, \mathscr{W}) = 3$.

Proof : By the strongness of $\mathscr{W}$, $\phi \neq \mathscr{W}$. Since $\mathscr{W}$ is proper, $\mathscr{V}(N, \mathscr{W}) \geq 3$. Suppose that for all $i \in N$ it holds $\{i\} \notin \mathscr{W}$. Let S be an $\subset$-minimal element of $\mathscr{W}$. By supposition, $|S| \geq 2$. Let $j \in S$. Then $S \backslash \{j\} \notin \mathscr{W}$. Since $\mathscr{W}$ is strong, I have $S' = (N \backslash S) \cup \{j\} \in \mathscr{W}$. Again by strongness of $\mathscr{W}$, $S'' = N \backslash \{j\} \in \mathscr{W}$. Then we have $\cap \{S, S', S''\} = \phi$. Thus $\mathcal{V}(N, \mathcal{W}) \leq 3$. ☺

Proof of Theorem 2 - 4 : Let F be a SWF satisfying (IIA), (UN) and $F(D) \subset W$. Then by Corollary 2 -5, $\mathcal{W}_F$ is proper and strong. Suppose that there does not exist $d \in N$ such that $\{d\} \in \mathcal{W}_F$. Then Lemma 2 - 6 implies $\mathcal{V}(N, \mathcal{W}) = 3$. Thus there exist three distinct winning coalitions such that the intersection of these is the empty set. But since $|X| \geq 3$, Lemma 1 - 1 and this together imply that F is cyclic. This is a contradiction. ☺

3. Concluding Remark

To the best of my knowledge, this paper is the first which derives Arrow's and Sen's theorems from a common lemma. Intuitively, this shows that these two theorems share a 'common source'. Although my derivation of Arrow's theorem is very akin to that of Wilson (1972), I think there are at least two additional merits in my method. Firstly, I made an explicit use of the theory of simple games in the proof of Arrow's theorem, and this proof intersects that of Sen's theorem in Lemma 1 - 1. As known to some readers, the theory of simple games enables us to discuss major results of the classical (i. e. finite population) Arrowian social choice theory from a unified viewpoint. (E. g. see Chapter 11 of Moulin (1988).) But Sen's theorem has been considered to be out of this 'unified viewpoint'. Indeed, since Sen's conditions do not give rise to winning coalitions, the theorem is not to be argued via simple games directly. However, Lemma 1 - 1 gives a 'bridge' which connects Sen's theorem to simple games in an indirect way. This connects Sen's theorem not only to Arrow's but also to various results to be treated from the 'unified viewpoint' provided by the concept of simple games.

The second point is educational. These two impossibility theorems are essential ingredients of any introductory course of microeconomics with normative perspective. But, putting aside the ethical implication, understanding these proofs is not-very-easy logical exercise for average undergraduate students. I think my method would be a nontrivial saving of energy to those students in studying these proofs.

Doctoral Student, Graduate School of Economics and Business Administration, Hokkaido University

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