Available online at www.sciencedirect.com
SCIENCE DIRECT®
Games and Economic Behavior 44 (2003) 379–389
GAMES and Economic Behavior
www.elsevier.com/locate/geb

The weak core of simple games with ordinal preferences: implementation in Nash equilibrium

Tomoichi Shinotsuka [a,*] and Koji Takamiya [b]

[a] *Department of Economics, Otaru University of Commerce, 3-5-21, Midori, Otaru 047-8501, Japan*
[b] *Institute of Social and Economic Research, Osaka University, 6-1, Mihogaoka, Ibaraki, Osaka 567-0047, Japan*

Received 11 January 2002

Abstract

In a simple game, coalitions belonging to a given class are "absolutely powerful" while others have no power. We attempt to make this distinction operational. Toward this end, we propose two axioms on social choice correspondences, Strong Non-Discrimination and Exclusion. Strong Non-Discrimination describes circumstances under which certain coalitions, the losing coalitions, have no influence over social choice. Exclusion requires that there are situations in which certain coalitions, the winning coalitions, can exercise their power. We show that the weak core correspondence is the minimal correspondence satisfying Maskin Monotonicity and Strong Non-Discrimination. We also show that the weak core is the unique correspondence satisfying Nash implementability, Strong Non-Discrimination, and Exclusion.

JEL classification: C71; C72; D71; D78

0. Introduction

This paper studies Nash-implementation of social choice correspondences (SCC for short) on the class of simple games with ordinal preferences. In a simple game, coalitions belonging to a given class are "absolutely powerful" while others have no decision power. In this paper, we attempt to make this distinction operational by proposing two axioms on social choice correspondences, Strong Non-Discrimination and Exclusion.

* Corresponding author.
E-mail address: shino@res.otaru-uc.ac.jp (T. Shinotsuka).

0899-8256/03/$ – see front matter
doi:10.1016/S0899-8256(03)00055-1

Strong Non-Discrimination describes circumstances under which certain coalitions, the "losing" coalitions, have no influence over social choice: for any such coalition and any two alternatives, if all the individuals outside the coalition are indifferent between these alternatives, then the SCC should not discriminate between them, in the sense that one alternative is chosen, i.e., contained in the image of the SCC, if and only if the other alternative is chosen. Strong Non-Discrimination is a strengthening of a condition used in different contexts (Thomson, 1987; Gevers, 1986; and Nagahisa, 1991, 1994). Exclusion requires that there are situations in which certain coalitions, the "winning" coalitions, can exercise their power: for any such coalition, if all the members of the coalition have identical, nontrivial preferences (in the sense that at least two alternatives are not judged indifferent), and, furthermore, all the members of the complementary coalition have trivial preferences (for them all the alternatives are indifferent), then the coalition has the power of excluding at least one alternative.

Some remarks are in order on Strong Non-Discrimination and Exclusion. The former cannot be interpreted as saying that losing coalitions have no power. In fact, it tells us nothing if some member of the complementary coalition finds one alternative preferable to another. Similarly, Exclusion demands that a SCC should grant a winning coalition a "right" to exclude some alternative only in the very special case when all of its members have common preferences and all the members of the complementary coalition regard all alternatives indifferent. Therefore, Exclusion does not seem to grant unlimited power to the winning coalitions. However, we will see that Strong Non-Discrimination and Exclusion become considerably stronger when the important axiom of Maskin Monotonicity is imposed as well.

Recall that a SCC satisfies Maskin Monotonicity if it preserves the desirability of an allocation under transformations of preferences that raise the relative ranking of the allocation. Maskin Monotonicity and Strong Non-Discrimination imply that losing coalitions have no veto power (Lemma 3.10). Furthermore, these three axioms together imply that winning coalitions are "all-powerful" and losing coalitions are "completely powerless" (Corollary 3.6).

Recall the distinction between the strong core and the weak core of simple games. An alternative is in the strong core of a simple game if there exist no winning coalition and another alternative that is at least as good for all members of the winning coalition and strictly preferred by some member of the winning coalition. An alternative is in the weak core if there exist no winning coalition and another alternative that is strictly preferred by all members of the winning coalition. The strong core satisfies Strong Non-Discrimination and Exclusion but violates Maskin Monotonicity (see Remark 3.7). Hence, the distinction between the weak core and the strong core is critical.

We show that the weak core correspondence is the minimal correspondence satisfying Maskin Monotonicity and Strong Non-Discrimination. This is the central result in this paper but, strictly speaking, it is not a full characterization of the weak core. Toward this end, we work with a domain on which the weak core is nonempty. Then, we show that the weak core is the unique correspondence satisfying Maskin Monotonicity, Strong Non-Discrimination, and Exclusion. It is well known that Maskin Monotonicity is necessary for Nash implementability. It turns out that the weak core is the unique correspondence satisfying Nash implementability, Strong Non-Discrimination, and Exclusion.[1]

Maskin monotonicity delivers a striking result to social choice functions (singleton-valued social choice correspondences). Indeed, Saijo (1987) showed that a social choice function satisfying a "dual dominance" condition is Maskin monotonic if and only if it is constant. As Saijo remarked, a social choice correspondence satisfies dual dominance if the domain of the correspondence contains the profile of trivial preferences. In this paper, we require that the domain of social choice correspondences satisfy a certain "richness" condition that implies that the profile of trivial preferences belongs to the domain. The weak core correspondence is Maskin monotonic but the weak core of the profile of trivial preferences is equal to the set of all alternatives. Hence the weak core correspondence is not singleton-valued. Therefore, Saijo's theorem does not apply to the weak core correspondence.

Implementability of the core has been investigated in different contexts.[2] Kara and Sönmez (1996) studied two-sided matching problems. They showed that the weak core correspondence is Nash implementable by means of Danilov's (1992) and Yamato's (1992) results. We employ the same technique to show that the weak core is Nash implementable in our framework, too. Unlike Kara and Sönmez, we can also apply Maskin's result on Nash implementability under the additional assumption that every singleton coalition is losing (Theorem 3.11). Sönmez (1996) studied generalized matching problems including housing markets and marriage problems. He generalized the results of Kara and Sönmez to the class of generalized matching problems.

Another strand of the literature deals with mechanisms implementing the core in economic environments. Wilson (1978) studied standard exchange economies. He proposed a two-stage, competitive bidding game in which one subgame perfect equilibrium outcome is always in the weak core of the exchange economy. The unfortunate part of Wilson's mechanism is that it only partially implements the weak core. Kalai et al. (1979) studied public good economies. They defined a strategic form mechanism that fully implements the weak core of the exchange economy in strong equilibrium. The mechanism, however, implements the individually rational correspondence in Nash equilibrium. Serrano and Vohra (1997) worked with private ownership economies including public goods. They constructed an extensive form mechanism that implements the weak core in subgame prefect equilibrium.

The plan of the paper is as follows: The next section provides preliminary definitions. Section 2 introduces the axioms for SCC's. Section 3 states the main results. And Section 4 gives concluding remarks.

---

[1] An anonymous referee suggested calling coalitions in the given class "privileged." Following this suggestion, we summarize our results as follows. Implementability implies that a "privileged" coalition must be "winning" (all-powerful) after all.

[2] For excellent surveys on implementation problems, see Maskin (1985), Moore (1992), and Jackson (2001).

1. Notations and definitions

We fix the set X of (*social*) *alternatives* (or *outcomes*), and the finite set N of *individuals*. Denote by $\Re$ the set of complete and transitive binary relations on X. Let $i \in N$, and $R^i \in \Re$. We denote by R^i individual i's *preference relation*. For $x, y \in X$, xR^iy means that to individual i, x is at least as good as y. We denote the asymmetric part of R^i by P^i, and the symmetric part by I^i. We say that $R^i \in \Re$ is *trivial* if individual i is indifferent among all alternatives, that is, for all $x, y \in X$, xI^iy. Denote by R_0^i individual i's trivial preference relation. We call $R = (R^i)_{i \in N} \in \Re^N$ a *preference profile*. The trivial preference profile R_0 consists of the trivial preference relation of each individual, i.e., $R_0 = (R_0^i)_{i \in N}$.

1.1. Simple games

Let W be a nonempty subset of $2^N - \{\phi\}$. We call a subset of N a coalition. A coalition in W is a *winning coalition* and a coalition not in W is a *losing coalition*. Let R be a preference profile. *For simplicity, we fix* W *from now on. Hence, we identify a simple game with a preference profile*. Let $x, y \in X$ and let $S \subset N$. We say that x *strongly dominates* y *via* S if S is winning and for all $i \in S$, xP^iy. We say that x *weakly dominates* y *via* S if S is winning, x is at least as good as y for every individual in S, and x is preferred by some individual in S. The *weak core of a simple game* R, denoted by $\mathrm{C}(R)$, is the set of all alternatives that are not strongly dominated by any other alternative. The *strong core of a simple game* R, denoted by $\mathrm{SC}(R)$, is the set of all alternatives that are not weakly dominated by any other alternative.

1.2. Implementation

For each $i \in N$, let D^i be a nonempty subset of $\Re$, interpreted as the set of *admissible preferences* of individual i. For a coalition S, D^S denotes the Cartesian product $\prod_{i \in S} D^i$. A *social choice correspondence* (henceforth, SCC) is a nonempty-valued correspondence $\varphi : D^N \to X$. We call D^N the *domain* of φ. For $R^i \in \Re$ and $x \in X$, let $L(R^i, x) = \{y \in X \mid xR^iy\}$. For preference profiles R and R', call R' a *monotonic transformation of* R *at* x if for all $i \in N$: $L(R^i, x) \subset L(R'^i, x)$. Denote by $\mathrm{MT}(R, x)$ the set of all monotonic transformations of R at x. Let φ be a SCC. In order to freely perform monotonic transformations, the domain D^N should satisfy a certain "richness" condition. That is, we say that D^N is *closed under monotonic transformations* (or simply, closed) if for all $R \in D^N$ and $x \in X$, $\mathrm{MT}(R, x) \subset D^N$. Under this assumption, the domain D^N contains the trivial profile R_0 because φ is nonempty-valued, and R_0 is a monotonic transformation of *any* R at *any* x.[3] A *game form* is a list (S, g) with $S = \prod_{i \in N} S^i$, where each S^i is a *strategy space* for individual i, and $g : S \to X$ is an *outcome function*. A *game* is a list (S, g, R), where (S, g) is a game form, and $R \in \Re^N$. A strategy profile $s \in S$ is a *Nash equilibrium of the game* (S, g, R) if there is no $i \in N$ such that for some $t^i \in S^i$: $g(t^i, s^{-i})P^ig(s)$. Denote by $\mathrm{NE}((S, g, R))$ the set of Nash equilibria of the game (S, g, R). Let φ be a SCC. Then

---

[3] The richness condition has another implication. See Appendix A.

the game form (S, g) *Nash-implements* φ if for all $R \in D^N$: $\varphi(R) = g(\mathrm{NE}((S, g, R)))$. Say that φ is *Nash-implementable* if there exists a game form which Nash-implements φ.

Necessary and sufficient conditions for Nash-implementation have been investigated in detail. A fundamental property is as follows. Let φ be a SCC.

Maskin Monotonicity (*MMON*): For all R and $R' \in D^N$, if $x \in \varphi(R)$ and $R' \in \mathrm{MT}(R, x)$, then $x \in \varphi(R')$.

Maskin (1999) proved that if a SCC is Nash-implementable, then it satisfies MMON. The converse of this claim does not hold true. Danilov (1992) and Yamato (1992) developed a sharper condition.[4] Let $i \in N$ and let $Y \subset X$. Let us denote $\mathrm{Ess}(i; \varphi; Y) = \{y \in Y \mid$ there exists $R \in D^N$ such that $y \in \varphi(R)$ and $L(R^i, y) \subset Y\}$.

Essential Monotonicity (*EMON*): For all R and $R' \in D^N$, if $x \in \varphi(R)$ and for all $i \in N$, $\mathrm{Ess}(i; \varphi; L(R^i, x)) \subset L(R'^i, x)$, then $x \in \varphi(R')$.

Yamato (1992) proved that

(i) if $|N| \geqslant 3$, any SCC satisfying EMON is Nash-implementable, and
(ii) under a certain mild condition imposed on admissible preferences, any Nash-implementable SCC satisfies EMON.

2. New axioms for social choice correspondences

In this section, we propose new axioms that describe how SCC's depend on the class of winning coalitions. Before we get to those, we start with a discussion of a standard axiom.

The first axiom says that if all individuals are indifferent between two alternatives, then the SCC should not treat the alternatives differently.

Non-Discrimination (*ND*): For all $R \in D^N$, $x, y \in X$, if for all $i \in N$, $x I^i y$, then $x \in \varphi(R)$ if and only if $y \in \varphi(R)$.

The axiom appears in axiomatic characterizations of the Walrasian correspondence (see Thomson (1987), Gevers (1986), and Nagahisa (1991, 1994)). We will see that ND is also useful in the context of axiomatic studies of the weak core of simple games (see Lemma 3.4). We also consider a strengthening of ND to obtain further results.

The next axiom says that if all the individuals outside a losing coalition find two alternatives indifferent, then the SCC should not treat the alternatives differently.

Strong Non-Discrimination (*SND*): For all $R \in D^N$, $x, y \in X$, and $S \subset N$, if for all $i \in N \backslash S$, $x I^i y$, and $S \notin \mathrm{W}$, then $x \in \varphi(R)$ if and only if $y \in \varphi(R)$.

[4] For other refined conditions of Maskin monotonicity, see, e.g., Maskin (1985), Williams (1986), Repullo (1987), Saijo (1988), McKelvey (1989), Moore and Repullo (1990), Sjöström (1991), and Ziad (1998).

In other words, the SCC treats two alternatives symmetrically, independently of the preferences of the members of the losing coalition, as long as all the members of the complementary coalition are indifferent between the alternatives. Hence, losing coalitions have no power then. However, SND is far from saying that losing coalitions are absolutely powerless. This is because SND is not applicable if two people outside a losing coalition have different preferences over the two alternatives.

The connection between ND and SND is straightforward. Since $\mathcal{W}$ does not contain the empty set, SND implies ND. The converse does not hold (for example, the Pareto correspondence violates SND but does satisfy ND).

The next axiom says that a winning coalition can exclude at least one alternative if all the members of the coalition have identical nontrivial preferences and all the members of the complementary coalition are indifferent among all the alternatives.

Exclusion (*EX*): For all $S \in \mathcal{W}$ and $R^S \in D^S$, if for all $i, j \in S$, $R^i = R^j$, and for all $i \in S$, $R^i \neq R_0^i$, then $\varphi(R^S, R_0^{-S}) \neq X$.

The axiom cannot be interpreted as saying that winning coalitions are all-powerful for the following reasons. First, excluding at least one alternative is not equivalent to picking the best alternative under the foregoing circumstances. Second, if some member of the coalition $N \backslash S$ has nontrivial preferences, EX does not apply.

3. Main results

In this section, we assume that every profile in the domain D^N of SCC's under consideration has a nonempty weak core and that D^N is closed under monotonic transformations. We start with the following important observation.

Lemma 3.1. *The weak core correspondence satisfies MMON.*

Proof. Let R, R' be two profiles and $x \in X$ be such that $R' \in \mathrm{MT}(R, x)$ and $x \notin \mathcal{C}(R')$. Thus under R', x is dominated by some other $y \in X$ via some $S \in \mathcal{W}$. That is, for all $i \in S$, $y P'^i x$. Since $R' \in \mathrm{MT}(R, x)$, this implies for all $i \in S$, $y P^i x$. Thus x is dominated by y under R. But this says that $x \notin \mathcal{C}(R)$. Therefore, $\mathcal{C}$ satisfies MMON. □

Theorem 3.2. *If a SCC φ satisfies MMON and SND, then φ is a supercorrespondence of the weak core $\mathcal{C}$, that is, for any $R \in D^N$, $\varphi(R) \supset \mathcal{C}(R)$.*

Proof. Let $x \in X$ and $R \in D^N$ be such that $x \in \mathcal{C}(R)$. Suppose that $x \notin \varphi(R)$. Let $y \in \varphi(R)$, $T^- = \{i \in N \mid x R^i y\}$, and $T^+ = \{i \in N \mid y P^i x\}$. Let R' be the profile obtained by raising y to the same indifference class as x and keeping the other indifference classes intact for each member of T^-: For all $i \in T^-$, $x I'^i y$ and for all $v, w \in X \backslash \{y\}$: $v R^i w$ if and only if $v R''^i w$ and $v I^i w$ if and only if $v I'^i w$, and for all $i \in T^+$, $R'^i = R^i$.

Note that $R' \in \mathrm{MT}(R, x)$. Since D^N is closed, $R' \in D^N$. By MMON, $y \in \varphi(R')$. Further, since $L(R^i, x) = L(R'^i, x)$ for each $i \in N$, we have $R \in \mathrm{MT}(R', x)$. Again, by

MMON, $x \notin \varphi(R')$. Thus φ treats x and y differently under R'. Suppose that T^+ is losing. Then, by SND either $x, y \in \varphi(R')$ or $x, y \notin \varphi(R')$, which is a contradiction. Hence, T^+ is winning. Thus under R', y dominates x via T^+. This says that $x \notin \mathrm{C}(R')$. Note that $R' \in \mathrm{MT}(R, x)$ since $L(R^i, x) = L(R'^i, x)$ for all $i \in N$. Lemma 3.1 tells us that C satisfies MMON. Thus $x \notin \mathrm{C}(R)$. This is a contradiction. □

Remark 3.3. In Theorem 3.2 above, the set inclusion may be strict. Consider the SCC that chooses the whole set of alternatives X for every profile. This SCC satisfies both MMON and SND.

Lemma 3.4. *If a SCC φ satisfies MMON, ND, and EX, then φ satisfies the following property*:

(★) *For all $S \subset N$, $R^S \in D^S$, and $x \in X$, if $S \in \mathrm{W}$ and for all $i \in S$, $y, z \in X \backslash \{x\}$, $x P^i y$ and $y I^i z$, then $\varphi(R^S, R_0^{-S}) = \{x\}$.*

Proof. Let $S \in \mathrm{W}$ and $x \in X$. Let R be a profile such that for all $i \in S$, $y, z \in X \backslash \{x\}$, $x P^i y$ and $y I^i z$, and for all $i \in N \backslash S$, $R^i = R_0^i$.

Let $R^* \in D^N$ and $v \in \varphi(R^*)$. Since $R_0 \in \mathrm{MT}(R^*, v)$ and D^N is closed, $R_0 \in D^N$. Then by MMON, $v \in \varphi(R_0)$. Since x is indifferent to v for any $i \in N$ at R_0, by ND $x \in \varphi(R_0)$. Note that $R \in \mathrm{MT}(R_0, x)$. Then since D^N is closed, $R \in D^N$. Then, by MMON, $x \in \varphi(R)$. On the other hand, by EX, $\varphi(R) \neq X$. That is, there is some alternative that does not belong to $\varphi(R)$. By construction, under R, any two alternatives other than x are indifferent for any member of N. Thus ND implies that any alternative other than x is excluded from $\varphi(R)$. □

Theorem 3.5. *If a SCC φ satisfies MMON, ND, and EX, then φ is a subcorrespondence of the weak core, that is, for any $R \in D^N$, $\varphi(R) \subset \mathrm{C}(R)$.*

Proof. Let $x \in X$ and $R \in D^N$. Assume that $x \in \varphi(R)$, but $x \notin \mathrm{C}(R)$. Then there exists $S \in \mathrm{W}$ and $y \in X$ such that for all $i \in S$, $y P^i x$. Let R' be a profile such that for all $i \in S$, $v, w \in X \backslash \{y\}$, $y P'^i v$ and $v I'^i w$, and for all $i \in N \backslash S$, $R'^i = R_0^i$.

Note that $R' \in \mathrm{MT}(R, x)$. Then by MMON, we have $x \in \varphi(R')$. By Lemma 3.4, $\varphi(R') = \{y\}$. This implies $x = y$, a contradiction. □

The following result is immediate from the foregoing theorems.

Corollary 3.6. W *is the unique SCC that satisfies MMON, SND, and EX.*

Remark 3.7. The above Corollary 3.6 characterizes the weak core of simple games. This result is tight in the sense that the three properties MMON, SND, and EX are logically independent. The example in Remark 3.3 establishes the independence of EX. For the independence of SND, fix $x \in X$ and let $\varphi(R) = \{x\}$ for all R. Then, φ obviously satisfies MMON and EX but violates SND. To show the independence of MMON, note that SC satisfies SND and EX but violates MMON. To illustrate a violation of MMON, let $X = \{x, y\}$, $N = \{1, 2\}$, $\mathrm{W} = \{N\}$, xP_1y, yP_2x, xI'_1y, yP'_2x. Then, $x \in \varphi(R)$, $R' \in \mathrm{MT}(R, x)$, but $x \notin \varphi(R')$.

Theorem 3.8. *Assume that* $|N| \geqslant 3$. *Then* C *is Nash-implementable.*

Proof. In view of Yamato (1992), it suffices to show that C satisfies EMON. Suppose otherwise. Then,

$(*)$ there exist $R, R' \in D^N$ such that $x \in \mathrm{C}(R)$ and for all $i \in N$, $\mathrm{Ess}(i; \mathrm{C}; L(R^i, x)) \subset L(R'^i, x)$ but $x \notin \mathrm{C}(R')$.

Since C satisfies MMON (Lemma 3.1), ($x \in \mathrm{C}(R)$ and $x \notin \mathrm{C}(R')$) in $(*)$ above implies that $R' \notin \mathrm{MT}(R, x)$. Hence, there exists $i \in N$ such that $L(R^i, x) \backslash L(R'^i, x) \neq \phi$. Let $z \in L(R^i, x) \backslash L(R'^i, x)$. By $(*)$, $z \notin L(R'^i, x)$ implies $z \notin \mathrm{Ess}(i; \mathrm{C}; L(R^i, x))$. Therefore, for some $i \in N$,

$(**)$ for all $R'' \in D^N$, if $L(R''^i, z) \subset L(R^i, x)$, then $z \notin \mathrm{C}(R'')$.

Now let R^* be a profile such that for all $j \in N$, $xI^{*j}z$ and for all v and $w \in X \backslash \{z\}$, $vR^{*j}w$ if and only if vR^jw.

Thus for each $j \in N$, we have $L(R^j, x) \cup \{z\} = L(R^{*j}, x)$. This implies $L(R^j, x) \subset L(R^{*j}, x)$. Thus, $R^* \in \mathrm{MT}(R, x)$. Since $x \in \mathrm{C}(R)$ and C satisfies MMON, we have $x \in \mathrm{C}(R^*)$. Since $L(R^i, x)$ contains z by assumption, $L(R^i, x) = L(R^{*i}, x)$. Since $xI^{*i}z$ by construction, $L(R^{*i}, z) = L(R^{*i}, x)$. Therefore $L(R^{*i}, z) = L(R^i, x)$. This equality and $(**)$ together imply $z \notin \mathrm{C}(R^*)$. On the other hand, $x \in \mathrm{C}(R^*)$ and for all $j \in N$, $xI^{*j}z$. Hence, $z \in \mathrm{C}(R^*)$. This is a contradiction. □

Now we conclude the following.

Theorem 3.9. *Assume that* $|N| \geqslant 3$. *Then* C *is the unique SCC satisfying Nash implementability, SND and EX.*

We discuss the logical relations between Maskin's '*no veto power*' condition and SND and EX. Let φ denote a SCC. Let S be a coalition. Say that S has *veto power* if there are some $R \in D^N$ and $x \in X$ such that $x \notin \varphi(R)$ and for all $j \in N \backslash S$, $L(R^j, x) = X$. Maskin (1999) introduced the following definition.

No Veto Power (*NVP*): For all $i \in N$, $R \in D^N$, and $x \in X$, if for all $j \in N \backslash \{i\}$, $L(R^j, x) = X$, then $x \in \varphi(R)$.

It directly follows from the definition that if a SCC satisfies EX, then every winning coalition has veto power.

Lemma 3.10. *If a SCC* φ *satisfies MMON and SND, then no losing coalition has veto power.*

Proof. Let φ satisfy MMON and SND. Let S be a losing coalition. Suppose that S has veto power. Then there are $R \in D^N$ and $x \in X$ such that for all $j \in N \backslash S$, $L(R^j, x) = X$ and $x \notin \varphi(R)$. Since φ satisfies SND, $S \notin \mathrm{W}$ implies $\varphi(R^S, R_0^{-S}) = X$. Since $L(R^j, x) = X = L(R_0^j, x)$, $R \in \mathrm{MT}((R^S, R_0^{-S}), x)$. Thus MMON implies $x \in \varphi(R)$. This contradicts $x \notin \varphi(R)$. □

Maskin (1999) proved that if a SCC satisfies both MMON and NVP, and $|N| \geqslant 3$, then it is Nash implementable. Combining Lemma 3.10 with Maskin's result, we obtain the following.

Theorem 3.11. *Assume $|N| \geqslant 3$. Let φ be a SCC. Assume that there does not exist $i \in N$ such that $\{i\} \in \mathrm{W}$. Then if φ satisfies MMON and SND, then φ is Nash-implementable.*

Remark 3.12. Theorems 3.2 and 3.11 immediately imply the following: Let there be at least three individuals. Assume there does not exist a winning coalition consisting of only one individual. Then if a SCC satisfies MMON and SND, then it is Nash implementable and it is a supercorrespondence of the weak core.

4. Concluding remarks

Though we require SCC's to be nonempty-valued, this requirement played little role in the foregoing arguments. Though the nonemptiness requirement has some conceptual appeal, there is a room for applying Occam's Razor. In fact, in axiomatic studies, some authors (see footnote 5, for example) do not insist on this requirement but explore how far they can go formally without it. In this section, we take this view.

In this section, we assume that the domain D^N contains at least one simple game R whose weak core is nonempty.

Let φ be a correspondence from D^N to X, possibly empty valued for some $R \in D^N$. We introduce the following property:

Restricted Nonemptiness (*RNEM*):[5] For all $R \in D^N$: $\mathrm{C}(R) \neq \phi$ implies $\varphi(R) \neq \phi$.

Theorem 4.1. *The weak core is the unique correspondence satisfying MMON, SND, EX, and RNEM.*

Proof. By the foregoing arguments, the weak core C satisfies these four properties. We shall prove uniqueness. First, it is easy to verify that Theorem 3.2 still holds even if the nonempty-valuedness of φ is weakened to the assumption that φ satisfies RNEM. This establishes $\varphi \supset \mathrm{C}$. Second, Theorem 3.5 is also true without nonempty-valuedness or RNEM. And in that case, nonempty-valuedness of the weak core C becomes superfluous.

[5] Similar properties can be found, for example, in Norde et al. (1996), and Peleg et al. (1996). These papers study axiomatizations of solutions to strategic form games.

Recall that SND implies ND. Then all conditions for Theorem 3.5 are met. This proves $\varphi \subset \mathrm{C}$. □

Remark 4.2. The properties appearing in Theorem 4.1 are logically independent. By adapting the examples in Remark 3.7 in a straightforward manner, we can show that each of MMON, SND, and EX is independent of the other three properties. The following example shows RNEM is independent of the other three properties: Let φ be the solution such that $\varphi(R) = \phi$ for any $R \in D^N$. Then φ satisfies MMON, SND, and EX but violates RNEM.

Acknowledgments

The first author is very grateful to William Thomson for helpful suggestions and comments, and to the faculty and staff members, especially Lionel W. McKenzie, of the Department of Economics at the University of Rochester for their hospitality. The second author thanks Hiroshi Ono for encouragement. The authors thank an associate editor of this journal and an anonymous referee for comments that greatly improved this paper. All errors are our own responsibility.

Appendix A

An anonymous referee pointed out an interesting implication of the richness condition that we imposed on the domain D^N. That is, if the domain D^N is closed under monotonic transformation, then it includes all preference profiles comprising preferences with top-ranked alternatives. Clearly, if the set of alternatives is finite, D^N is unrestricted. The following proof is due to the referee.

Let $R' = (R'^1, \ldots, R'^n)$ be any arbitrary profile comprising of preferences with top-ranked alternatives, and let, for each i, x^i denote the top-ranked alternative for individual i. We know that the trivial profile R_0 belongs to D^N. Let $(R'^1, R_0^{-\{1\}})$ be a profile such that only player 1 has a (possibly) nontrivial preference R'^1. Since $(R'^1, R_0^{-\{1\}})$ is a monotonic transformation of R_0 at x^1, $(R'^1, R_0^{-\{1\}})$ belongs to D^N. Similarly, $(R'^1, R'^2, R_0^{-\{1,2\}})$ is a monotonic transformation of $(R'^1, R_0^{-\{1\}})$ at x^2, so $(R'^1, R'^2, R_0^{-\{1,2\}})$ belongs to D^N. By induction hypothesis, $(R'^1, R'^2, \ldots, R'^{n-1}, R_0^{-\{1,2,\ldots,n-1\}})$ belongs to D^N. Since $R' = (R'^1, \ldots, R'^n)$ is a monotonic transformation of $(R'^1, R'^2, \ldots, R'^{n-1}, R_0^{-\{1,2,\ldots,n-1\}})$ at x_n, it belongs to D^N.

References

Danilov, V., 1992. Implementation via Nash equilibria. Econometrica 60, 43–56.

Gevers, L., 1986. Walrasian social choice: some simple axiomatic approaches. In: Heller, W.P., Starr, R., Starret, D. (Eds.), Social Choice and Public Decision Making: Essays in Honor of K.J. Arrow, Vol. 1. Cambridge Univ. Press, London.

Jackson, M.O., 2001. A crash course on implementation theory. Soc. Choice Welfare 18, 655–708.

Kalai, E., Postlewaite, A., Roberts, J., 1979. A group incentive compatible mechanism yielding core allocations. J. Econ. Theory 20, 13–22.

Kara, T., Sönmez, T., 1996. Nash implementation of matching rules. J. Econ. Theory 68, 425–439.

Maskin, E., 1999. Nash equilibrium and welfare optimality. Rev. Econ. Stud. 66, 23–38.

Maskin, E., 1985. The theory of implementation in Nash equilibrium: a survey. In: Hurwicz, L., Schmeidler, D., Sonnenschein, H. (Eds.), Social Goals and Social Organizations. Cambridge Univ. Press, Cambridge.

McKelvey, R.D., 1989. Game forms for Nash implementation of general social choice correspondence. Soc. Choice Welfare 6, 139–156.

Moore, J., 1992. Implementation, contracts and renegotiation in environments with complete information. In: Laffont, J.J. (Ed.), Advances in Economic Theory: Sixth World Congress of the Econometric Society, Vol. 1. Cambridge Univ. Press, Cambridge.

Moore, J., Repullo, R., 1990. Nash implementation: a full characterization. Econometrica 58, 1083–1099.

Nagahisa, R.I., 1991. A local independence condition for characterization of Walrasian allocation rule. J. Econ. Theory 54, 106–123.

Nagahisa, R.I., 1994. A necessary and sufficient condition for Walrasian social choice. J. Econ. Theory 62, 186–208.

Norde, H., Potters, J., Reijnierse, H., Vermeulen, D., 1996. Equilibrium selection and consistency. Games Econ. Behav. 12, 219–225.

Peleg, B., Potters, J.A.M., Tijs, S.H., 1996. Minimality of consistent solutions for strategic games, in particular for potential games. Econ. Theory 7, 81–93.

Repullo, R., 1987. A simple proof of Maskin's theorem on Nash implementation. Soc. Choice Welfare 6, 139–156.

Saijo, T., 1988. Strategy space reduction in Maskin's theorem: sufficient conditions for Nash implementation. Econometrica 56, 693–700.

Saijo, T., 1987. On constant Maskin monotonic social choice functions. J. Econ. Theory 42, 382–386.

Serrano, R., Vohra, R., 1997. Non-cooperative implementation of the core. Soc. Choice Welfare 14, 513–525.

Sjöström, T., 1991. On the necessary and sufficient conditions for Nash implementation. Soc. Choice Welfare 8, 333–340.

Sönmez, T., 1996. Implementation in generalized matching problems. J. Math. Econ. 26, 429–439.

Thomson, W., 1987. The vulnerability to manipulate behavior of resource allocation mechanisms designed to select equitable and efficient outcomes. In: Groves, T., Radner, R., Reiter, S. (Eds.), Information, Incentives, and Economic Mechanisms: Essays in Honor of Leonid Hurwicz. Univ. Minnesota Press, Minneapolis.

Williams, S., 1986. Realization and Nash implementation: two aspects of mechanism design. Econometrica 54, 139–151.

Wilson, R., 1978. Competitive exchange model. Econometrica 46, 577–585.

Yamato, T., 1992. On Nash implementation of social choice correspondences. Games Econ. Behav. 4, 484–492.

Ziad, A., 1998. A new necessary condition for Nash implementation. J. Math. Econ. 29, 381–387.