Coalition Strategy-Proofness and Monotonicity in Shapley–Scarf Housing Markets*

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Abstract

This paper studies (single-valued) solutions to housing markets (Shapley and Scarf, 1974) with strict preferences. I show that a solution is monotonic if and only if it is coalition strategy-proof. I point out that the strong core solution is the only solution which is monotonic, individually rational and an onto function. (As Roth and Postlewaite (1977) showed, the strong core solution is single-valued when the preferences are strict.) This result follows from the above equivalence theorem and a preceding characterization of the strong core solution by Ma (1994). My characterization sharpens Sönmez's (1996) similar result by weakening Pareto optimality to ontoness. I also provide some related results. A solution is strategy-proof and nonbossy if and only if it is monotonic. Thus the strong core solution is the unique solution which is strategy-proof, nonbossy, individually rational, and onto.

JEL Classification— C71, C78, D71, D78.

Keywords— Shapley-Scarf housing market, coalition strategy-proofness, monotonicity, equivalence theorem.

0 Introduction

This paper examines single-valued solutions to housing markets by an axiomatic method. A *housing market* consists of finitely many agents each of whom is endowed with one unit of perfectly indivisible good (say a house). Each agent has a preference over the goods. A *single-valued solution* is a function which specifies one allocation of the indivisible goods for each preference profile. Since in this paper I consider only single-valued solutions, I refer to them simply as solutions.

Housing markets were introduced by Shapley and Scarf (1974). They examined the core, the strong core and competitive equilibria of housing markets. They proved the nonemptiness of the core and the existence of competitive equilibria for any preference profiles, and that the strong core may be empty if weak preferences are admissible (i.e.

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an agent can be indifferent between two goods.) However, Roth and Postlewaite (1977) showed that if every agent has a strict preference, then the strong core is always a singleton, and coincides with the unique competitive equilibrium. In this paper, I basically postulate strict preferences. Specifically, I investigate the strong core solution (or the strong core, simply) as a relevant solution.

The strong core satisfies various 'desirable' properties. Among others, here I am particularly interested in *monotonicity* and *strategy-proofness* properties. Strategy-proofness properties are incentive compatibility requirements. A solution is *strategy-proof* if the solution is immune to individual manipulations, i.e. no single agent can be better off by misrepresenting his preference. Also I consider situations where a solution is even robust to manipulations by groups. In this case, according to the strength of the definition of coalitional manipulation, two nonmanipulability concepts are defined: *weak coalition strategy-proofness*, and *coalition strategy-proofness*. Roth (1982) proved that the strong core is strategy-proof. Furthermore, by Bird (1984), the strong core is even coalition strategy-proof (thus also weakly coalition strategy-proof). Ma (1994) proved that the strong core is the unique solution which is strategy-proof, individually rational and Pareto optimal.

In this paper, a solution being *monotonic* means that the chosen allocation does not vary by a 'monotonic' change of the preference profile. It is well known that similar monotonicity properties play significant roles in implementation theory (see e.g. Maskin, 1985). The strong core is monotonic in this sense. Sönmez (1996) showed that the strong core is the only solution which is monotonic, individually rational and Pareto optimal.[1]

I question how these strategy-proofness properties and monotonicity are related to each other. My motivation is famous results from studies on voting rules[2] : The Gibbard–Satterthwaite theorem asserts that if a voting rule (with at least three alternatives) is an onto function, then the rule is strategy-proof if and only if it is dictatorial (Gibbard, 1973; Satterthwaite, 1975). Muller and Satterthwaite (1977) proved that for voting rules, monotonicity is equivalent to strategy-proofness. This says that an onto voting rule with at least three alternatives is dictatorial if and only if it is monotonic. These results motivate to ask whether there are analogous relations between monotonicity and strategy-proofness in housing markets.

Answering to this question, I prove that *monotonicity is equivalent to coalition strategy-proofness*. And combining the above result with a preceding characterization of the strong core by Ma (1994), I show that *the strong core is the unique solution which is monotonic, individually rational and an onto function*. This characterization sharpens Sönmez's (1996) result mentioned above by weakening Pareto optimality to ontoness.

I also provide some additional results in connection with the well-known axiom 'nonbossiness'. I point out that *a solution is strategy-proof and nonbossy if and only if it is monotonic*. This result implies another new characterization: *the strong core is the only solution that is strategy-proof, nonbossy, individually rational and onto*.

Throughout the paper, though basically assuming strict preferences, I examine how the results change when weak preferences are admissible.

The plan of this paper is as follows: The next section provides definitions. Section 2 states the main results. Section 3 discusses the additional results. Section 4 proves independences of the axioms used in the characterizations of the strong core. Finally,

[1] Sönmez (1996) showed this result as a corollary to his theorem that deals with the Nash implementation problem in a more general environment than housing markets.

[2] In this paper, a voting rule means a function (called 'social choice function') specifying one alternative (candidate) for each preference profile of a finite number of individuals (voters). For each individual, precisely all the strict preferences are admissible (the unrestricted domain assumption). I followed Moulin (1988) on this terminology.

Section 5 concludes the paper.

1 Model

A *housing market* is a triple (N, H, R). Here $N = \{1, \ldots, n\}$ with $n \geq 2$ is the set of agents. Each agent i owns an indivisible good w^i. Then $H := \{w^i\}_{i \in N}$ is the set of goods. $R = (R^i)_{i \in N}$ is a profile of preference relations on H. Any R^i is assumed to be complete (i.e. $h, k \in H \Rightarrow (hR^ik$ or $kR^ih))$ and transitive (i.e. $(h, k, m \in H$ & hR^ik & $kR^im) \Rightarrow hR^im)$. For two indivisible goods $h, k \in H$, hR^ik reads that to agent i h is at least as good as k. P^i and I^i denote the asymmetric part and the symmetric part of R^i, respectively. Thus hP^ik means agent i prefers h to k, whereas hI^ik denotes the indifference between h and k. Call R^i *strict* if R^i is anti-symmetric (i.e. $(h, k \in H$ & $hI^ik) \Rightarrow h = k)$. And R^i is *weak* if R^i is either strict or not. In what follows, I assume preference relations are all strict unless noted. Denote the set of strict preferences on H by $\mathscr{P}$. For $S \subset N$, denote $\mathscr{P}^S := \prod_{i \in S} \mathscr{P}^i$ with $\mathscr{P}^i = \mathscr{P}$. [3] An *allocation* is a bijection $x : N \to H$. Here, $x(i)$ is the good allocated to agent i. Denote the set of allocations by $\mathscr{A}$. Let the sets N and H be given. A *single-valued solution* is a function $\varphi : \mathscr{P}^N \to \mathscr{A}$. Thus for $i \in N$ and $R \in \mathscr{P}^N$, $\varphi(R)(i)$ denotes the good allocated to agent i. In the following, I call a single-valued solution simply a *solution*.

Let a housing market (N, H, R) be given. Let $x, y \in \mathscr{A}$ and $S \subset N$ with $S \neq \emptyset$. Then say that x *weakly dominates* y via S if $\{w^i \in H \mid i \in S\} = \{x(i) \in H \mid i \in S\}$ & $(\forall i \in S : x(i)R^iy(i)$ & $\exists j \in S : x(j)P^jy(j))$. An allocation x is in the *strong core* if there does not exist any other allocation which weakly dominates x.[4] It is known that the strong core is a singleton for each $R \in \mathscr{P}^N$, and coincides with the (unique) competitive allocation[5] (Roth and Postlewaite, 1977). I refer to the solution which specifies the strong core allocation for each $R \in \mathscr{P}^N$ as the *strong core solution* (*strong core*, for short).

I examine the following properties (axioms) of a solution φ. First, I introduce three versions of strategy-proofness.

Strategy-Proofness (SP): $\forall i \in N : \forall R \in \mathscr{P}^N : \forall R'^i \in \mathscr{P} : \varphi(R)(i)R^i\varphi(R^{-i}, R'^i)(i)$.

This axiom (SP) says that no single agent can be strictly better off by misrepresenting his preference.

Weak Coalition Strategy-Proofness (wCSP): $\forall S \subset N, S \neq \emptyset : \forall R \in \mathscr{P}^N : \forall R'^S \in \mathscr{P}^S : (\forall i \in S : \varphi(R^{-S}, R'^S)(i)R^i\varphi(R)(i)) \Rightarrow (\exists j \in S : \varphi(R^{-S}, R'^S)(j)I^j\varphi(R)(j))$.

This axiom (wCSP) implies (SP). This says that no group of agents can collusively misrepresenting their preferences in a way that each member of the group will be strictly better off.

Coalition Strategy-Proofness (CSP): $\forall S \subset N, S \neq \emptyset : \forall R \in \mathscr{P}^N : \forall R'^S \in \mathscr{P}^S : (\forall i \in S : \varphi(R^{-S}, R'^S)(i)R^i\varphi(R)(i)) \Rightarrow (\forall i \in S : \varphi(R^{-S}, R'^S)(i)I^i\varphi(R)(i))$.

[3] Inclusion '$\subset$' is weak.

[4] Say that x *dominates* y via S if $\{w^i \in H \mid i \in S\} = \{x(i) \in H \mid i \in S\}$ & $(\forall i \in S : x(i)P^iy(i))$. This defines the *core* in the same way (see Shapley and Scarf, 1974).

[5] A *competitive allocation* is an allocation x which satisfies for some price vector $(p^i)_{i \in N}$ (p^i is the price of the good w^i), $\forall i, j \in N : x(i) = w^j \Rightarrow p^j \leq p^i$ (budget constraints), and $w^jP^ix(i) \Rightarrow p^i < p^j$ (utility maximization).

This axiom (CSP) implies (wCSP). (CSP) means that no group of agents can collusively misreport their preferences in a way that some member of the group will be better off without any other member being worse off. The strong core satisfies all these three strategy-proofness axioms ((SP), (wCSP) and (CSP)) (Roth, 1982; Bird, 1984). The following two axioms are standard.

Individual Rationality (IR): $\forall R \in \mathscr{P}^N : \forall i \in N : \varphi(R)(i) R^i w^i$.

Pareto Optimality (PO): $\forall R \in \mathscr{P}^N : \forall x \in \mathscr{A} : (\forall i \in N : x(i) R^i \varphi(R)(i)) \Rightarrow (\forall i \in N : x(i) I^i \varphi(R)(i))$.

It is immediate that the strong core satisfies the axioms (IR) and (PO). Also the strong core satisfies the following axiom.

Ontoness (ONTO): $\forall x \in \mathscr{A} : \exists R \in \mathscr{P}^N : \varphi(R) = x$.

The axiom (ONTO) says that φ is an onto function. That is , no allocation is excluded a priori. Note that the axiom (PO) implies (ONTO).

Next, I introduce monotonicity. Let $R^i \in \mathscr{P}$ and $h \in H$. Then denote by $L(h, R^i)$ the set $\{k \in H \mid h R^i k\}$.

Monotonicity (MON): $\forall R, R' \in \mathscr{P}^N : (x = \varphi(R) \,\&\, (\forall i \in N : L(x(i), R^i) \subset L(x(i), R'^i))) \Rightarrow x = \varphi(R')$.

Note that since N is a finite set, the axiom (MON) is equivalent to the following condition:

$$\forall R \in \mathscr{P}^N : \forall i \in N : R'^i \in \mathscr{P} : (x = \varphi(R) \;\&\; L(x(i), R^i) \subset L(x(i), R'^i)) \Rightarrow x = \varphi(R^{-i}, R'^i).$$

(MON) says that the if an allocation is chosen, then that allocation will still be chosen if each agent changes his preference keeping or improving the relative ranking of the good allocated to him. The strong core satisfies (MON). This follows from the equivalence between (CSP) and (MON) (to be proved in Theorem 2.1) and the fact that the strong core satisfies (CSP) (Bird, 1984). Essentially the same property can be defined also for voting rules.[6] (MON) is a version of ‘Maskin monotonicity’, which plays an important role in implementation theory (see e.g. Maskin, 1977; 1985).

2 Main Results

The following is the main result. Let φ denote a solution.

Theorem 2.1 *φ satisfies (MON) if and only if φ satisfies (CSP).*

Proof *(If part)* Let φ satisfy (CSP). Suppose that φ does not satisfy (MON). Then $\exists i \in N : \exists R \in \mathscr{P}^N : \exists R'^i \in \mathscr{P} : (x = \varphi(R) \;\&\; L(x(i), R^i) \subset L(x(i), R'^i)) \;\&\; x \neq \varphi(R^{-i}, R'^i)$. Let $\varphi(R^{-i}, R'^i) = y$. And denote $C := \{j \in N \mid y(j) \neq x(j)\}$. Clearly, C is not empty.

Case 1: Assume $i \in C$. Then either $y(i) P^i x(i)$ or $x(i) P^i y(i)$ is true. Suppose that the former is true. Then I have $\varphi(R^{-i}, R'^i)(i) P^i \varphi(R)(i)$. That is, agent i gains from misreporting his preference as R'^i instead of R^i. Thus this contradicts (CSP). And suppose that the latter is true. Then $x(i) P'^i y(i)$. Thus I have $\varphi(R)(i) P'^i \varphi(R^{-i}, R'^i)(i)$. That is, agent i gains from misreporting his preference as R^i instead of R'^i. Again, a contradiction to (CSP).

[6]Muller and Satterthwaite (1977) called that property ‘strong positive association’.

Case 2: Assume $i \notin C$. Then $\varphi(R^{-i}, R'^i)(i) = \varphi(R)(i)$. Define $D := \{j \in C \mid y(j)P^j x(j)\}$ and $E := \{j \in C \mid x(j)P^j y(j)\}$. Note that $C = D \cup E$ and $D \cap E = \emptyset$.

(i) Assume $D \neq \emptyset$. Let $S = D \cup \{i\}$. Let $R^{\star S} = (R^{S-\{i\}}, R'^i) \in \mathscr{P}^S$. Then I have

$$\forall j \in S - \{i\} : \varphi(R^{-S}, R^{\star S})(j) P^j \varphi(R)(j), \text{ and } \varphi(R^{-S}, R^{\star S})(i) = \varphi(R)(i).$$

This says that all the members of S but agent i improve without agent i's loss by a collusive misrepresentation of their preferences as $R^{\star S}$ instead of R^S. That is, a contradiction to (CSP).

(ii) Assume $D = \emptyset$. Then $E \neq \emptyset$ since $C \neq \emptyset$. Let $T = E \cup \{i\}$. Let $R^{\star\star T} = (R^{T-\{i\}}, R'^i) \in \mathscr{P}^T$. Note that $\forall j \in T - \{i\} : R^{\star\star j} = R^j$. Then I have

$$\forall j \in T - \{i\} : \varphi(R^{-T}, R^T)(j) P^{\star\star j} \varphi(R^{-T}, R^{\star\star T})(j) \;\&\; \varphi(R^{-T}, R^T)(i) = \varphi(R^{-T}, R^{\star\star T})(i).$$

This says that all the members of T but agent i improve without agent i's loss by a collusive misrepresentation of their preferences as R^T instead of $R^{\star\star T}$. Hence, a contradiction again to (CSP).

(Only if part) Let φ satisfy (MON). Suppose that φ does not satisfy (CSP). Then $\exists S \subset N,\ S \neq \emptyset : \exists R \in \mathscr{P}^N : \exists R'^S \in \mathscr{P}^S$:

$$(\forall i \in S : \varphi(R^{-S}, R'^S)(i) R^i \varphi(R)(i)) \;\&\; (\exists j \in S : \varphi(R^{-S}, R'^S)(j) P^j \varphi(R)(j)).$$

Let $x = \varphi(R)$ and $y = \varphi(R^{-S}, R'^S)$. Note that $x \neq y$. Let $T = \{i \in S \mid y(i)P^i x(i)\}$. Thus for $i \in S - T$ (notice that $S - T$ may be empty), $y(i) = x(i)$. Let $R^{\star S}$ be any element of $\mathscr{P}^S$ which satisfies[7]

$$\forall i \in T : \text{top}R^{\star i}(H) = y(i) \;\&\; \text{top}R^{\star i}(H - \{y(i)\}) = x(i), \text{ and}$$
$$\forall i \in S - T : \text{top}R^{\star i}(H) = x(i) = y(i).$$

Then I have $\forall i \in S : L(x(i), R^i) \subset L(x(i), R^{\star i})$. Recall that $x = \varphi(R)$. Thus (MON) implies that $\varphi(R^{-S}, R^{\star S}) = x$. Also notice that $\forall i \in S : L(y(i), R'^i) \subset L(y(i), R^{\star i})$. Recall that $y = \varphi(R^{-S}, R'^S)$. Hence by (MON), I have $\varphi(R^{-S}, R^{\star S}) = y$. Thus I conclude that $x = y$. This is a contradiction. □

Remark 2.2 As mentioned, for voting rules, strategy-proofness is equivalent to monotonicity (Muller and Satterthwaite, 1977). But in housing markets, even the axiom (wCSP) is too weak to imply (MON). To see this, consider a solution ψ defined as follows:

$$\forall R \in \mathscr{P}^N :$$
$$\psi(R)(1) = \text{top}R^1(H),$$
$$\psi(R)(2) = \text{top}R^1(H - \{\psi(R)(1)\}),$$
$$\psi(R)(3) = \text{top}R^1(H - \{\psi(R)(1), \psi(R)(2)\}),$$
$$\ldots \text{ proceeding inductively.}$$

Clearly, ψ satisfies (SP) and (wCSP), but not (CSP). (Note that the allocation $\psi(R)$ is determined independent of R^j with $j \neq 1$.)

Remark 2.3 Since it is shown that (CSP) is strictly stronger than (wCSP) in the above remark, one should ask about the logical relation between (SP) and (wCSP). For voting rules, as shown in Ishikawa and Nakamura (1979), strategy-proofness is equivalent to coalition strategy-proofness. (In this case, coalition strategy-proofness and weak coalition

[7] Let R^i be a preference relation. Let $Y \subset H$. Then I denote $\text{top}R^i(Y) := \{h \in Y \mid \forall k \in Y : hR^i k\}$. If $\text{top}R^i(Y)$ is a singleton, then I slightly abuse the notation and denote by $\text{top}R^i(Y)$ the single element.

strategy-proofness are the same for the assumption of the strict preference domain. Recall Footnote 2.) But this is not the case for housing markets. The following example shows that (SP) does not imply (wCSP). Let n be an even integer with $n \geq 4$. Let $m = n/2$. Then let $S_1 = \{1, \ldots, m\}$ and $S_2 = \{m+1, \ldots, n\}$. Note that $\{S_1, S_2\}$ is a partition of the set of agents N, and that $|S_1| = |S_2|$. And correspondingly, the set of goods H is partitioned into $\{H_1, H_2\}$, where $H_p := H|_{S_p} = \{w^i\}_{i \in S_p}$ with $p = 1, 2$. Here and in the sequel, '$|$' denotes restriction. Then I obtain two 'split' housing markets from these partitions. For $p = 1, 2$, let $\sigma_p : \mathscr{P}|_{H_p}{}^{S_p} \to \mathscr{A}|_{S_p}$ be the strong core of the housing market consisting of the agent set S_p and the set of goods H_p. Here the set of allocations $\mathscr{A}|_{S_p}$ is the set of bijections from S_p onto H_p. Then define a solution ψ to the housing market consisting of N and H as follows:

$$\begin{aligned}
&\forall R \in \mathscr{P}^N : \\
&\forall i \in S_1 : \psi(R)(i) = \sigma_2((R^j|_{H_2})_{j \in S_2})(m+i), \text{ and} \\
&\forall i \in S_2 : \psi(R)(i) = \sigma_1((R^j|_{H_1})_{j \in S_1}))(i-m).
\end{aligned}$$

In the above solution, an allocation within subgroup S_1 is determined by the preferences of subgroup S_2 over H_2, and vice versa. Thus it is obvious that ψ satisfies (SP). But ψ does not satisfy (wCSP).

Remark 2.4 [8] If I assume weak preferences, then the 'if' part of Theorem 2.1 does not hold. I have the following counterexample:

Let $N = \{1, 2\}$. Hence $H = \{w^1, w^2\}$. Let $\mathscr{R}$ denote the set of weak preferences on H. (Note that $\mathscr{P} \subset \mathscr{R}$.) Let ψ be a solution such that $\forall R \in \mathscr{R}^N$:

$$\begin{aligned}
&\psi(R) = (\text{the strong core allocation for } R) \text{ if } R \in \mathscr{P}^N, \\
&\psi(R)(i) = \text{top} R^i(H) \text{ and } \psi(R)(j) = H - \{\psi(R)(i)\} \text{ if } R^i \in \mathscr{P} \text{ and } R^j \notin \mathscr{P}, \text{ and} \\
&\psi(R)(i) = w^i \ \forall i \in N, \text{ otherwise.}
\end{aligned}$$

ψ satisfies (CSP). The following proves this. Clearly, ψ satisfies (PO). This implies that the group $\{1, 2\}$ does not gain from misreporting. Now suppose that a single agent i gains from misrepresentation when his true preference is R^i, and the other agent j reports R^j. Then it must be that $R^i \in \mathscr{P}$ and $\psi(R)(j) P^i \psi(R)(i)$. This implies $R^j \in \mathscr{P}$. Then since (PO) is satisfied, I have $\psi(R)(j) P^j \psi(R)(i)$. Thus it follows that $(\psi(R)(i), \psi(R)(j)) = (w^i, w^j)$. Now it is easy to check that agent i alone cannot change the resulting allocation by shifting his preference reporting. This is a contradiction.

ψ does not satisfy (MON). There is an example to check this. Let R^1 and R^2 satisfy $w^1 I^i w^2$ for $i = 1, 2$. Then $\psi(R) = (w^1, w^2)$. But when R'^2 is such that $w^1 P'^2 w^2$ (note that $H = L(w^1, R^2) = L(w^1, R'^2)$), I have $\psi(R^1, R'^2) = (w^2, w^1) \neq \psi(R)$. Thus the 'if' part fails to be true. However, the 'only if' part still holds true (i.e.(MON) implies (CSP)) under weak preferences. The proof applies without essential change.

Lemma 2.5 *If φ satisfies (CSP) and (ONTO), then φ satisfies (PO).*

Proof Let φ satisfy (CSP) and (ONTO). Suppose that φ does not satisfy (PO). Then $\exists R \in \mathscr{P}^N : \exists x \in \mathscr{A} : (\forall i \in N : x(i) R^i \varphi(R)(i))$ & $(\exists j \in N : x(j) P^j \varphi(R)(j))$. (ONTO) implies $\exists R' \in \mathscr{P}^N : x = \varphi(R')$. Then I have $\exists R \in \mathscr{P}^N : \exists R' \in \mathscr{P}^N : (\forall i \in N : \varphi(R')(i) R^i \varphi(R)(i))$ & $(\exists j \in N : \varphi(R')(j) P^j \varphi(R)(j))$. This says that φ violates (CSP). This is a contradiction. □

[8] Here and in the sequel, abusing the notation, for a solution φ, $\varphi(R)$ denotes the n-tuple $(\varphi(R)(i))_{i=1}^n$.

Remark 2.6 Lemma 2.5 still holds true even if I assume weak preferences. The proof is essentially the same.

The strong core can be characterized by strategy-proofness. The following result was established by Ma (1994).

Theorem 2.7 (Ma, 1994) *φ is the strong core if and only if φ satisfies (SP), (IR) and (PO).*

Proof See Ma (1994).

Combining my results and Ma's theorem in the above, I obtain the following characterization of the strong core.

Corollary 2.8 *φ is the strong core if and only if φ satisfies (MON), (IR) and (ONTO).*

Proof *(If part)* Immediate from Theorem 2.1, Lemma 2.5 and Theorem 2.7.

(Only if part) It is immediate that the strong core satisfies (IR) and (ONTO). Since the strong core satisfies (CSP) (Bird, 1984), Theorem 2.1 implies that it satisfies (MON). $\square$

Remark 2.9 Sonmez (1996) proved that the strong core is the only solution that satisfies (MON), (IR) and (PO). Corollary 2.8 sharpens this result by weakening (PO) to (ONTO). Sonmez (1996) showed this result as a corollary to his theorem that deals with the Nash implementation problem in a more general environment than housing markets.

3 Further Results

1. Since it has been proved that the axiom (MON) is equivalent to the axiom (CSP), it may be questioned if there exists a monotonicity property which is equivalent to the axioms (SP) or (wCSP). I have a partial answer. I introduce the following definition.

Individual Monotonicity (IMON): $\forall R \in \mathscr{P}^N : \forall i \in N : \forall R'^i \in \mathscr{P} : (x = \varphi(R) \ \& \ L(x(i), R^i) \subset L(x(i), R'^i)) \Rightarrow x(i) = \varphi(R^{-i}, R'^i)(i)$.

Lemma 3.10 *Let φ be a solution. Then φ satisfies (IMON) if and only if φ satisfies (SP).*

Proof The proof is very similar to that of Theorem 2.1.

(If part) Let φ satisfy (SP). Suppose that φ does not satisfy (IMON). Then $\exists i \in N : \exists R \in \mathscr{P}^N : \exists R'^i \in \mathscr{P} : (x = \varphi(R) \ \& \ L(x(i), R^i) \subset L(x(i), R'^i)) \ \& \ x(i) \neq \varphi(R^{-i}, R'^i)(i)$. Then either $\varphi(R^{-i}, R'^i)(i) P^i \varphi(R)(i)$ or $\varphi(R)(i) P^i \varphi(R^{-i}, R'^i)(i)$ is true. If the former is true, then it contradicts (SP). If the latter is true, then I also have $\varphi(R)(i) P'^i \varphi(R^{-i}, R'^i)(i)$, which contradicts (SP).

(Only if part) Let φ satisfy (IMON). Suppose that φ does not satisfy (SP). Then $\exists i \in N : \exists R \in \mathscr{P}^N : \exists R'^i \in \mathscr{P} : \varphi(R^{-i}, R'^i)(i) P^i \varphi(R)(i)$. Let us denote $x = \varphi(R)$ and $y = \varphi(R^{-i}, R'^i)$. Thus I have $x(i) \neq y(i)$. Then let $R^{\star i}$ be any element of $\mathscr{P}$ which satisfies $\text{top}R^{\star i}(H) = y(i)$ and $\text{top}R^{\star i}(H - \{y(i)\}) = x(i)$. Note that $L(x(i), R^i) \subset L(x(i), R^{\star i})$. By (IMON), this implies $\varphi(R^{-i}, R^{\star i})(i) = x(i)$. Also note that $L(y(i), R'^i) \subset L(y(i), R^{\star i})$. By (IMON), this implies $\varphi(R^{-i}, R^{\star i})(i) = y(i)$. Thus I have $x(i) = y(i)$. This is a contradiction. $\square$

(IMON) is obviously implied by (MON). It says that if an agent keeps or improves the relative ranking of the good allocated to him (fixing the others' preferences), then the good allocated to him (rather than the allocation itself) does not change.

Remark 3.11 The 'only if' part of this lemma holds true on the weak preference domain $\mathscr{R}^N$. But the 'if' part does not. A counterexample is the solution ψ in Remark 2.4, which satisfies (SP) but not (IMON).

2. I examine the results obtained in relation to the following well-known axiom (Satterthwaite and Sonnenschein, 1981).

Nonbossiness (NB): $\forall R \in \mathscr{P}^N : \forall i \in N : \forall R'^i \in \mathscr{P} : \varphi(R^{-i}, R'^i)(i) = \varphi(R)(i) \Rightarrow \varphi(R^{-i}, R'^i) = \varphi(R)$.

The axiom (NB) says that when an agent changes his preference reporting, he cannot influence the total allocation without affecting his own allocation. The following fact is straightforward.

Theorem 3.12 *φ satisfies (MON) if and only if φ is (SP) and (NB).*

Proof *(If part)* By definition, (IMON) and (NB) together imply (MON). Thus by Lemma 3.10, the desired conclusion is obtained.

(Only if part) By Theorem 2.1, (MON) implies (SP). Now I show that (MON) also implies (NB). Suppose that φ satisfies (MON) but not (NB). Then $\exists R \in \mathscr{P}^N : \exists i \in N : \exists R'^i \in \mathscr{P} : \varphi(R^{-i}, R'^i)(i) = \varphi(R)(i)$ & $\varphi(R^{-i}, R'^i) \neq \varphi(R)$. Denote $\varphi(R)$ by x. Then choose a preference relation $R^{\star i}$ of agent i such that $x(i) = \text{top}R^{\star i}(H)$. Then Clearly, $L(x(i), R^i) \subset L(x(i), R^{\star i})$ and $L(x(i), R'^i) \subset L(x(i), R^{\star i})$. Thus (MON) implies $\varphi(R^{-i}, R^{\star i}) = \varphi(R)$ and $\varphi(R^{-i}, R^{\star i}) = \varphi(R^{-i}, R'^i)$. Thus I have $\varphi(R^{-i}, R'^i) = \varphi(R)$, a contradiction. □

Svensson (1999) has already pointed out that [(SP) & (NB)] $\Rightarrow$ (MON), but not the converse. Analogous relations are observed in other models (e.g. Barberà and Jackson, 1995).

Remark 3.13 When the domain is the weak preferences $\mathscr{R}^N$, the 'if' part of Theorem 3.12 does not hold true. To see this, consider the following solution ψ. Let $R \in \mathscr{R}^N$. Consider a construction $R \mapsto R_\star,\ R_\star \in \mathscr{P}^N$ as to satisfy the following:

$$\forall i \in N : \forall j, k \in N : [[j > k \ \& \ w^j I^i w^k] \text{ or } w^j P^i w^k] \Rightarrow w^j P^i_\star w^k.$$

Note that this construction is unique. Let us denote by σ the strong core. Then define a solution ψ such that

$$\forall R \in \mathscr{R}^N : \psi(R) = \sigma(R_\star).$$

That is to say, the solution ψ is the strong core with 'tie-breaking according to the indexes of the goods'. This solution ψ satisfies (SP) and (NB) but not (CSP).[9] Recall that as noted in Remark 2.4, (MON) still implies (CSP) on the domain of weak preferences $\mathscr{R}^N$. Thus ψ does not satisfy (MON).

[9]This solution ψ is discussed in Bird (1984). He shows by example that ψ satisfies (SP) but not (CSP).

Remark 3.14 The 'only if' part of Theorem 3.12 still holds true on the domain $\mathscr{R}^N$. The proof is essentially the same. But I note that on this domain (CSP), which is equivalent to (MON) on the domain $\mathscr{P}^N$, does not imply (NB), although on the strict preference domain $\mathscr{P}^N$, this implication is trivially true. A counterexample is as follows: Let $N = \{1, 2, 3\}$. Assume that the domain is $\mathscr{R}^N$. Let $\{\mathscr{Q}, \mathscr{S}\}$ be a partition of $\mathscr{R}$. Define a solution ψ as to satisfy the following: Let $R \in \mathscr{R}^N$.

(1) If $w^1 I^1 w^2$ & $w^1 I^2 w^2$, then

   (i) if $R^3 \in \mathscr{Q}$, then $\psi(R) = (w^1, w^2, w^3)$, and

   (ii) if $R^3 \in \mathscr{S}$, then $\psi(R) = (w^2, w^1, w^3)$.

(2) Otherwise, $\psi(R) = (\sigma^1, \sigma^2, w^3)$.

Here (σ^1, σ^2) coincides with the strong core allocation of the housing market $(\{1, 2\}, \{w^1, w^2\}, R|_{\{w^1,w^2\}})$. Note that (σ^1, σ^2) is unique for any R in the case (2) above. Then ψ satisfies (CSP) but not (NB) (since agent 3 changes the total allocation fixing his own).

Remark 3.15 Summing up the results obtained, the following four sets of axioms are all logically equivalent: {(SP), (NB)}, {(IMON), (NB)}, {(CSP)} and {(MON)}. However, when the domain of a solution is extended to the weak preferences, only the equivalence [(IMON) & (NB)] $\Leftrightarrow$ (MON) is preserved. By the examples in Remarks 2.4, 3.11, 3.13 and 3.14, the other pairwise equivalences are all invalidated. (Note that the solution in Remark 3.11 satisfies (NB).)

Now the following characterization is immediate.

Corollary 3.16 *φ is the strong core if and only if φ satisfies (SP), (NB), (IR) and (ONTO).*

4 Independences of the Axioms

I show the logical independences of the axioms in each of the two characterizations of the strong core: [(MON), (IR) and (ONTO)] and [(SP), (NB), (IR) and (ONTO)]. Since I have (MON) $\Leftrightarrow$ [(SP) & (NB)] (see Theorem 3.12), it suffices to show the independences for the system [(SP), (NB), (IR) and (ONTO)] by means of examples. Let σ denote the strong core in the following.

Example 4.17 Let ψ be a solution such that

$$\forall R \in \mathscr{P}^N : \forall i \in N : \psi(R)(i) = w^i.$$

Then ψ satisfies (SP), (NB) (thus (MON)) and (IR), but not (ONTO).

Example 4.18 Let ψ be a solution such that [10]

$$\begin{aligned}
&\forall R \in \mathscr{P}^N : \\
&\psi(R)(1) = \text{top}R^1(H), \\
&\psi(R)(2) = \text{top}R^2(H - \{\psi(R)(1)\}), \\
&\psi(R)(3) = \text{top}R^3(H - \{\psi(R)(1), \psi(R)(2)\}), \\
&\ldots \text{ proceeding inductively.}
\end{aligned}$$

Then ψ satisfies (SP), (NB) (thus (MON)) and (ONTO), but not (IR).

[10] This solution is an example of simple serial dictatorship (Abdulkadiroğlu and Sönmez, 1998; Svensson, 1999).

Example 4.19 Assume that $n \geq 3$. Let $R \in \mathscr{P}^N$. Consider a construction $R \mapsto R_\star$, $R_\star \in \mathscr{P}^N$ such that

$$\forall i \in N : [h, m \in L(w^i, R^i) \;\&\; hP^i m] \Rightarrow hP^i_\star m, \text{ and}$$
$$[j < k \;\&\; w^j, w^k \in H - L(w^i, R^i)] \Rightarrow w^j P^i_\star w^k.$$

Let ψ be a solution which satisfies

$$\forall R \in \mathscr{P}^N : \psi(R) = \sigma(R_\star)$$

Then ψ satisfies (NB), (IR) and (ONTO), but not (SP) (thus nor (MON)).

Example 4.20 Assume that $n = 4$. Define $Q = \{R \in \mathscr{P}^N \mid (w^2 P^1 w^1 P^1 w^3 P^1 w^4) \;\&\; (w^1 P^2 w^2 P^2 w^3 P^2 w^4) \;\&\; (w^4 P^3 w^3) \;\&\; (w^3 P^4 w^4)\}$. Let ψ be a solution such that

$$\psi(R) = (w^2, w^1, w^3, w^4) \text{ if } R \in \mathscr{Q}, \text{ and}$$
$$\psi(R) = \sigma(R), \text{ otherwise.}$$

Note that for $R \in \mathscr{Q}$, $\sigma(R) = (w^2, w^1, w^4, w^3)$. Then ψ satisfies (SP), (IR) and (ONTO), but not (NB) (thus nor (MON)).

5 Conclusion

This paper has investigated logical relations among coalition strategy-proofness, monotonicity and other related axioms for solutions to housing markets with strict preferences. I showed that the following four sets of axioms are all logically equivalent: {(SP), (NB)}, {(IMON), (NB)}, {(CSP)} and {(MON)}. Two new characterizations of the strong core has been obtained: A solution is the strong core if and only if it satisfies [(MON), (IR) and (ONTO)] or [(SP), (NB), (IR) and (ONTO)].

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